

Definable C^r fiber bundle structures of a definable fiber bundle

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Received July 7, 2017

Abstract

Let G and K be definably compact subgroups of orthogonal groups and $0 \leq r < \infty$. We prove that every definable fiber bundle over an affine definable C^r manifold whose structure group is K admits a unique strongly definable C^r fiber bundle structure up to definable C^r fiber bundle isomorphism.

2010 *Mathematics Subject Classification*. 14P10, 14P20, 57S10, 57S15, 58A05, 58A07, 03C64.

Keywords and Phrases. O-minimal, definable groups, definable C^r groups, definable fiber bundles, definable C^r fiber bundles, definable G vector bundles, definable $C^r G$ vector bundles.

1. Introduction.

J. Bochnak, M. Coste and M.F. Roy proved that the following theorem.

Theorem 1.1 (12.7.14. [1]). *Every semialgebraic vector bundle over an affine Nash manifold admits a unique strongly Nash vector bundle structure up to Nash vector bundle isomorphism.*

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (R, +, \cdot, <)$ of a real closed field.

Everything is considered in \mathcal{N} and the term “definable” is used throughout in the sense of “definable with parameters in \mathcal{N} ”, each definable map is assumed to be continuous.

General references on o-minimal structures are [3], [4], also see [17].

In this paper we prove the definable fiber bundle version of the above result.

Theorem 1.2. *Let $\eta = (E, p, X, F, K)$ be a strongly definable fiber bundle over an affine definable C^r manifold and K an affine definably compact definable C^r group.*

(1) *There exists a strongly definable C^r fiber bundle ζ over X such that ζ is fiber bundle isomorphic to η .*

(2) *If ζ' is another strongly definable C^r fiber bundle over X such that ζ' is fiber bundle isomorphic to η , then ζ' and ζ are definably C^r fiber bundle isomorphic.*

In particular, (1) and (2) say that η admits a unique definable C^r fiber bundle structure up to definable C^r fiber bundle isomorphism. \square

If $R = \mathbb{R}$, then Theorem 1.1 is proved in [7].

Theorem 1.3. *Let G be a finite group and $0 \leq r < \infty$. Every definable G vector bundle over an affine definable $C^r G$ manifold admits a unique strongly definable $C^r G$ vector bundle structure up to definable $C^r G$ vector bundle isomorphism.*

2. Proof of our results.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \rightarrow Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A group G is a *definable group* if G is a definable set and the group operations $G \times G \rightarrow G$ and $G \rightarrow G$ are definable. A definable subset X of R^n is *definably compact* if for every definable map $f : (a, b)_R \rightarrow X$, there exist the limits $\lim_{x \rightarrow a+0} f(x), \lim_{x \rightarrow b-0} f(x)$ in X , where $(a, b)_R = \{x \in R \mid a \leq x < b\}, -\infty \leq a < b \leq \infty$. A definable subset X of R^n is *definably compact* if and only if X is closed and bounded ([16]). Note that if X is a definably compact definable set and $f : X \rightarrow Y$ is a definable map, then $f(X)$ is definably compact.

If R is the field of real numbers \mathbb{R} , then for any definable subset X of \mathbb{R}^n , X is compact if and only if it is definably compact. In general, a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} \mid 0 \leq x \leq 1\}$ is definably compact but not compact.

A *definable space* is an object obtained by pasting finitely many definable sets together along definable open subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [3]). Definable spaces are generalizations of semialgebraic spaces in the sense of [2].

Recall the definition of definable fiber bundles [12].

Definition 2.1. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable

space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \rightarrow X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \rightarrow U_i \times F\}_i$ of η such that each U_i is a definable open subset of X , $\{U_i\}_i$ is a finite open covering of X . For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \rightarrow F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \rightarrow F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \rightarrow K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\bar{f} : E \rightarrow E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) The map \bar{f} covers a definable map, namely there exists a definable map $f : X \rightarrow X'$ such that $f \circ p = p' \circ \bar{f}$.

(b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \bar{f} \circ \phi_{i,x}^{-1} : F \rightarrow F$ lies in K , and $f_{ij} : U_i \cap f^{-1}(V_j) \rightarrow K$ is a definable map.

We say that a bijective definable fiber bundle morphism $\bar{f} : E \rightarrow E'$ is a *definable fiber bundle equivalence* if it covers a definable homeomorphism $f : X \rightarrow X'$ and $(\bar{f})^{-1} : E' \rightarrow E$ is a definable fiber bundle morphism

covering $f^{-1} : X' \rightarrow X$. A definable fiber bundle equivalence $\bar{f} : E \rightarrow E'$ is called a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$.

- (3) A continuous section $s : X \rightarrow E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i , the map $\phi_i \circ s|_{U_i} : U_i \rightarrow U_i \times F$ is a definable map.
- (4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K . We write (E, p, X, K) for (E, p, X, F, K) .

A *definable C^r manifold* is a Hausdorff space with a finite system of charts whose transition functions are definable, and definable C^r maps, definable C^r diffeomorphisms and definable C^r imbeddings are defined similarly ([13], [10]). A definable C^r manifold is *affine* if it is definably C^r imbeddable into some \mathbb{R}^n . If $\mathcal{M} = \mathbb{R}$, a definable C^ω manifold (resp. affine definable C^ω manifold) is called a *Nash manifold* (resp. an *affine Nash manifold*). By [11], if $R = \mathbb{R}$ and $0 \leq r < \infty$, then every definable C^r manifold is affine. The definable C^ω case is complicated. Even if $\mathcal{M} = \mathbb{R}$, it is known that for every compact or compactifiable C^ω manifold of positive dimension admits a continuum number of distinct nonaffine Nash manifold structures (IV.1.3 [18]), and its equivariant version is proved in [14].

A *definable $C^r G$ action* on a definable C^r manifold X is a group action $G \times X \rightarrow X$ such that it is a definable C^r map.

Recall the definition of definable C^r fiber bundles [10].

Definition 2.2 ([10]). (1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable C^r fiber bundle* if the total space E and the base space X are definable C^r manifolds, the structure group K is a definable C^r group, the fiber F is a definable $C^r K$ manifold with an effective action, the projection

p is a definable C^r map and all transition functions of η are definable C^r maps. A *principal definable C^r fiber bundle* is defined similarly.

- (2) *Definable C^r fiber bundle morphisms, definable C^r fiber bundle equivalences, definable C^r fiber bundle isomorphisms* between definable C^r fiber bundles and *definable C^r sections* of a definable C^r fiber bundle are defined similarly.

Recall existence of definable quotient.

Theorem 2.3. (*Existence of definable quotient (e.g. 10. 2.18 [3])*). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \rightarrow X/G$ is surjective, definable and definably proper.

By a similar proof of 2.10 [15] and Theorem 2.3, we have the following.

Proposition 2.4. Let (E, p, X, K) be a principal definable fiber bundle, F a definable set with an effective definable K action and K a definably compact definable group. Then $(E \times_K F, p', X, F, K)$ is a definable fiber bundle, where $p' : E \times_K F \rightarrow X$ denotes the projection defined by $p'([z, f]) = p(z)$.

We have the definable C^r version of Proposition 2.4 similarly.

Proposition 2.5. Let (E, p, X, K) be a principal definable C^r fiber bundle over a definable C^r manifold X , F an affine definable C^r manifold with an effective definable $C^r K$ action and K an affine definably compact definable C^r group. Then $(E \times_K F, p', X, F, K)$ is a definable C^r fiber bundle, where $p' : E \times_K F \rightarrow X$ denotes the projection defined by $p'([z, f]) = p(z)$.

As a corollary of Proposition 2.5, we have the following proposition.

Proposition 2.6. Let $\mathcal{B}_K = (B_K, p_K, X_K)$ be the n -universal principal bundle relative to K , F an affine definable C^r manifold with an effective definable $C^r K$ action and K an affine definably compact definable C^r group. Then the associated fiber bundle $\mathcal{B}_K[F] := (E, p, X_K, F, K)$ is a definable C^r fiber bundle.

Theorem 2.7 ([5]). Let $X \subset R^n, Y \subset R^m$ be definable C^r manifolds and $0 \leq s < r < \infty$. Every definable C^s map $f : X \rightarrow Y$ is approximated by a definable C^r map $h : X \rightarrow Y$ in the definable C^s topology.

Definition 2.8. (1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is strongly definable if there exist the n -universal bundle \mathcal{B}_K and a definable map $f : X \rightarrow X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to η .

(1) A definable C^r fiber bundle $\eta = (E, p, X, F, K)$ is strongly definable if there exist the n -universal bundle \mathcal{B}_K and a definable C^r map $f : X \rightarrow X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably C^r fiber bundle isomorphic to η .

Proof of Theorem 1.2. (1) Since η is strongly definable, there exists the n -universal bundle \mathcal{B}_K and a definable map $f : X \rightarrow X_K$ such that $f^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to η . By Theorem 2.7, we have a definable C^r map $h : X \rightarrow X_K$ as an approximation of f . In particular h is definably homotopic to f . Thus by 1.1 [8], $\zeta := h^*(\mathcal{B}_K[F])$ is definably fiber bundle isomorphic to $f^*(\mathcal{B}_K[F])$ and ζ is a strongly definable C^r fiber bundle.

(2) Let ζ' be another strongly definable C^r fiber bundle over X such that ζ' is definably fiber bundle isomorphic to η . Consider the strongly definable C^r fiber bundle (ζ, ζ', id_X) whose sections represent the fiber bundle isomorphisms between ζ and ζ' which is defined in 2.11 [7]. Then it has a continuous section. By a way similar to the proof of 2.12 [7], it admits a definable C^r section. This section gives a definable C^r fiber bundle isomorphism between ζ and ζ' . \square

Definition 2.9. Let G be a definable C^r group and $0 \leq r < \infty$. Let Ω be an n -dimensional representation of G and let B be the representation map $G \rightarrow O_n(R)$ of Ω . Suppose that $M(\Omega)$ denotes the vector space of $n \times n$ -matrices with the action $(g, A) \in G \times M(\Omega) \rightarrow B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k , we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, A = A', Tr A = k\}$, $E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega | Av = v\}$, $u : E(\Omega, k) \rightarrow G(\Omega, k) : u((A, v)) = A$, where A' denotes the transposed matrix of A and $Tr A$ stands for the trace of A . Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G vector bundle. We call it the universal G vector bundle associated with Ω and k . Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets.

Definition 2.10. (1) Let G be a definable group. A definable G vector bundle $\eta = (E, p, X)$ over a definable G set X is called *strongly definable* if there exist a representation Ω of G and a definable G map $f : X \rightarrow G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η .

(2) Let G be a definable C^r group and $0 \leq r \leq \infty$. A definable $C^r G$ vector bundle $\eta = (E, p, X)$ over an affine definable $C^r G$ manifold X is called *strongly definable* if there exist a representation Ω of G and a definable $C^r G$ map $f : X \rightarrow G(\Omega, k)$ such that η is definably $C^r G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η .

To consider an equivariant version of Theorem 2.7, we need the averaging function.

Let $G = \{g_1, \dots, g_m\}$, X an affine definable $C^r G$ manifold and Ω a representation of G . Then we define the averaging function $A : C^r(X, \Omega) \rightarrow C^r(X, \Omega)$ by $A(f)(x) = \frac{1}{m} \sum_{i=1}^m g_i^{-1} f(g_i x)$.

Then we have the following proposition.

Proposition 2.11. (1) If f is a definable C^r map, then $A(f)$ is a definable $C^r G$

map.

(2) Let $\text{Def}^r(X, \Omega)$ (resp. $\text{Def}_G^r(X, \Omega)$) denote the set of definable C^r maps (resp. definable $C^r G$ maps) from X to Ω . Then $A|\text{Def}_G^r(X, \Omega) = \text{id}_{\text{Def}_G^r(X, \Omega)}$ and $A(\text{Def}^r(X, \Omega)) = \text{Def}_G^r(X, \Omega)$.

(3) $A : \text{Def}^r(X, \Omega) \rightarrow \text{Def}^r(X, \Omega)$ is continuous in the definable C^r topology.

As in the proof of 1.2 [9], we have the following result.

Proposition 2.12. *Let G be a finite group, X a definable $C^r G$ submanifold of a representation Ω of G and $1 \leq r < \infty$. Then there exists a definable $C^r G$ tubular neighborhood (U, θ) of X in Ω , namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \rightarrow X$ is a definable $C^r G$ map such that $\theta|_X = \text{id}_X$.*

By Theorem 2.7, Proposition 2.11 and 2.12, we have the following theorem.

Theorem 2.13. *Let G be a finite group and X, Y definable $C^r G$ submanifolds of representations Ω, Ξ of G , respectively and $0 \leq s < r < \infty$. Every definable $C^s G$ map $f : X \rightarrow Y$ is approximated by a definable $C^r G$ map with respect to the definable C^s topology.*

Proof of Theorem 1.3. Let η be a strongly definable G vector bundle over X . Since η is strongly definable, there exists a definable G map $f : X \rightarrow G(\Omega, \alpha)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, \alpha))$.

By Theorem 2.13, f is approximated by a definable $C^r G$ map $h : X \rightarrow G(\Omega, \alpha)$. By a way to the proof of 1.7 [10], η is definably G vector bundle isomorphic to a strongly definable $C^r G$ vector bundle $h^*(\gamma(\Omega, \alpha))$.

Let ζ be another strongly definable $C^r G$ vector bundle which is definably G vector bundle isomorphic to η . Since η and ζ are strongly definable $C^r G$ vector bundles, as in the proof of 3.1 [6], $\text{Hom}(\eta, \zeta)$ is a strongly definable $C^r G$ vector bundle. Since η and ζ are definably G vector bundle isomorphic, this definable G vector bundle isomorphism

defines a definable G section of $\text{Hom}(\eta, \zeta)$. Using Theorem 2.13, by a way similar to [6], η and ζ are definably $C^r G$ vector bundle isomorphic. \square

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